MATH 512, FALL 14 COMBINATORIAL SET THEORY WEEK 2

Definition 1. *A cardinal κ is measurable of there exists a non-principal, κ-complete ultrafilter on κ.*

Note that if U is such an ultrafilter, then every $X \in U$ has size κ. (Otherwise X would be the union of less than κ-many singletons.) Similarly, for every $\gamma < \kappa$, $\kappa \setminus \gamma \in U$.

Lemma 2. *Suppose that κ is measurable. Then κ is inaccessible.*

Proof. Let U be a non-principal, κ-complete ultrafilter on κ. If κ were singular, then for some $\tau < \kappa$, $\kappa = \subset_{i<\tau} X_i$ where every $|X_i| < \kappa$. But then by κ-completeness, for some i, $X_i \in U$. Contradiction. So κ is regular.

To show string limit, suppose that for some $\tau < \kappa$, $2^\tau \geq \kappa$. Let $\{A_\eta \mid \eta < \kappa\}$ be distinct subsets of τ with $\bigcup_{\eta<\kappa} A_\eta = \tau$. For every $\alpha < \tau$, define:

- $X^+_\alpha := \{\eta < \kappa \mid \alpha \in A_\eta\}$;
- $X^-_\alpha := \{\eta < \kappa \mid \alpha \notin A_\eta\}$.

For each α, one of these is in U, so let X_α be that set. By κ-completeness, $X := \bigcap \alpha < \tau X_\alpha \in U$. Let $A := \{\alpha < \tau \mid X^+_\alpha \in U\}$, and let $\eta \in X$. Then:

- if $\alpha \in A$, then $\eta \in X^+_\alpha$, and so $\alpha \in A_\eta$, and
- if $\alpha \notin A$, then $\eta \in X^-_\alpha$, and so $\alpha \notin A_\eta$.

So $A = A_\eta$. But similarly, if $\delta \in X$, then $A = A_\delta = A_\eta$, $\delta = \eta$. So, X is a singleton. Contradiction. □

Lemma 3. *Suppose that κ is measurable, then there is a non-principal, κ-complete, normal ultrafilter on κ.*

Proof. Let U be a non-principal, κ-complete ultrafilter on κ. For functions $f, g : \kappa \to \kappa$, let $f <_U g$ mean that $\{\alpha \mid f(\alpha) < g(\alpha)\} \in U$. Since U is κ-complete, we have that $<_U$ is well-founded. So let f be $<_U$-minimal, such that for all $\gamma < \kappa$, $\{\alpha \mid \gamma < f(\alpha)\} \in U$. Define $D := \{X \subset \kappa \mid f^{-1}(X) \in U\}$. It is routine to check that D is a non-principal, κ-complete ultrafilter. We will show normality.

Suppose that $A \in D$ and $h : A \to \kappa$ is regressive. Let $g(\alpha) = h(f(\alpha))$ if $\alpha \in A$ and $g(\alpha) = 0$ otherwise. Then $g_U f$, so there is some $\gamma < \kappa$, such that $\{\alpha \mid \gamma \geq g(\alpha)\} \in U$. Then by κ-completeness of U, there is some $\beta \leq \gamma$, such that $\{\alpha \mid \beta = g(\alpha)\} \in U$. Then $\{\alpha \in A \mid h(\alpha) = \beta\} \in D$.

□

Lemma 4. *κ is a measurable iff there is an elementary nontrivial embedding $j : V \to M$ with critical point κ and $M^\kappa \subset M$.*

Proof. For the first direction, if U is a measure on κ, let $j : V \to Ult(V, U)$ be given by $j(x) = [c_x]_U$. Here c_x is the constant function with value x. Then, $\phi(x_1, ..., x_n)$ holds in V iff $\{\alpha \mid \phi(c_{x_1}(\alpha), ..., c_{x_n}(\alpha))\} = \kappa \in U$ iff (by Los), $M \models \phi(x(x_1), ...j(x_n))$. So, j is elementary. Also since U is κ-complete, the ultrapower $Ult(V, U)$ is wellfounded, so we can identify it with its transitive collapse.

Claim 5. *For all $\alpha < \kappa$, $j(\alpha) = \alpha$.*

Proof. By induction on $\alpha < \kappa$. Suppose that $\beta < \kappa$, and for all $\alpha < \beta$, $j(\alpha) = [c_\alpha]_U = \alpha$. If $\beta = \alpha + 1$, by elementarity, $j(\beta) = j(\alpha) + 1 = \alpha + 1$. Suppose that β is limit. First, for every $\alpha < \beta$, $\alpha = [c_\alpha] <_U [c_\beta]_U$, so $\beta \leq [c_\beta]_U$. Also, if $[f]_U < [c_\beta]_U$, then $\{\gamma \mid f(\gamma) < \beta\} \in U$. Since U is κ-complete, for some $\alpha < \beta$, $\{\gamma \mid f(\gamma) = \alpha\} \in U$, i.e. $[f]_U = [c_\alpha]_U$. So $[c_\beta]_U = \sup_{\alpha<\beta}[c_\alpha] = \sup_{\alpha<\beta} \alpha = \beta$.

□

Let $id : \kappa \to \kappa$ be given by $id(\alpha) = \alpha$ for all α.

Claim 6. $\kappa \leq [id] < j(\kappa)$.

Proof. If $\gamma < \kappa$, then by the above claim $\gamma = [c_\gamma] < [id]$. The latter is because $\{\alpha < \kappa \mid \gamma > \alpha\} \in U$. So, $\kappa = \sup_{\gamma<\kappa} \gamma = \sup_{\gamma<\kappa}[c_\gamma]_U \leq [id]$. Also, since $\{\alpha < \kappa \mid id(\alpha) < c_\kappa(\alpha)\} = \kappa \in U$, we get $[id] < j(\kappa)$.

□

It follows that the critical point of j is κ. The last thing to show is that $M^\kappa \subset M$. For that, suppose that $\langle [f_i] \mid i < \kappa \rangle$ is a sequence of elements in M. Let $f : \kappa \to V$ be $f(\alpha) = \langle f_i(\alpha) \mid i < \alpha \rangle$. Then $[f] \in M$. Also since every $f(\alpha)$ is a sequence of length α, by Los's theorem, in M, $[f]$ is a sequence of length κ. Also, in V, $\{\alpha < \kappa \mid$ the $i-$th element of $f(\alpha) = f_i(\alpha)\} = \{\alpha < \kappa \mid i < \alpha\} \in U$. So, by Los, the i-th element of the sequence $[f]$ is exactly $[f_i]$. It follows that $[f] = \langle [f_i] \mid i < \kappa \rangle \in M$.

For the other direction, given an embedding $j : V \to M$, let $U := \{X \subset \kappa \mid \kappa \in j(X)\}$. It is straightforward by elementarity to check that U is an ultrafilter. For κ-completeness: suppose that $\tau < \kappa$ and $\langle X_\alpha \mid \alpha < \tau \rangle$ are sets in U. Then for every α, $\kappa \in j(X_\alpha)$. Since $j(\tau) = \tau$, we have that $j(\langle X_\alpha \mid \alpha < \tau \rangle) = \langle j(X_\alpha) \mid \alpha < \tau \rangle$. Then $\kappa \in \bigcup_{\alpha<\tau} j(X_\alpha) = j(\bigcup_{\alpha<\tau} X_\alpha)$, and so $\bigcup_{\alpha<\tau} X_\alpha \in U$.

For normality, suppose that $f : \kappa \to \kappa$ is a regressive function, i.e. $X = \{\alpha \mid f(\alpha) < \alpha\} \in U$. Then $\kappa \in j(X)$, and so $jf(\kappa) < \kappa$. It follows that for some $\gamma < \kappa$, $jf(\kappa) = \gamma$. Let $Y := \{\alpha \mid f(\alpha) = \gamma\}$. Then $\kappa \in j(Y)$, which means that $Y \in U$.

□

If U is a measure on κ, j_U denotes the embedding $x \mapsto [c_x]_U$. Since the ultrapower is well founded, we will identify $Ult(V, U)$ with its transitive collapse M, and write $j = j_U : V \to M \simeq Ult(V, U)$ for the embedding.

Lemma 7. *Suppose that κ is measurable and $j : V \to M \simeq Ult(V, U)$ is an elementary embedding as above. Then $\kappa = [id]$ iff U is normal.*

Proof. We already saw that $\kappa \leq [id]$. Then $\kappa = [id]$ iff for all $[f] < [id]$, there is some $\gamma < \kappa$, such that $[f] = \gamma$ iff for all regressive functions $f : \kappa \to \kappa$, there is some γ, such that $\{\alpha < \kappa \mid f(\alpha) = \gamma\} \in U$ iff U is normal. □

Note that this implies that if U is a normal measure, and $j = j_U : V \to M \simeq Ult(V, U)$, then $U = \{X \subset \kappa \mid \kappa \in j(X)\}$.

Lemma 8. *If κ is measurable and U is a normal measure on κ, then:*

- *U extends the club filter i.e. every club is in U, and*
- *if $X \in U$, then X is stationary.*

Proof. Let U be a normal measure on κ, and $j : V \to M \simeq Ult(V, U)$ be the embedding obtained from U. If C is a club, then $j"C = C \in M$ is unbounded in κ. So, in M, $j(C)$ is a club in $j(\kappa)$, and κ is a limit point of $j(C)$. It follows that $\kappa \in j(C)$, and so $C \in U$. For the second assertion, suppose that $X \in U$. Then for any club $C \subset \kappa$, $\kappa \in j(C) \cap j(X)$. So by elementarity $X \cap C \neq \emptyset$. □

Lemma 9. *If κ is measurable, then κ is Mahlo.*

Proof. Let $j : V \to M$ be an elementary embedding with critical point κ, and $M^\kappa \subset M$. Let $Reg := \{\alpha < \kappa \mid \alpha \text{ is regular}\}$. Since κ is regular in V and $M^\kappa \subset M$, then $M \models \kappa$ is regular. So, $\kappa \in j(Reg)$, and so $Reg \in U$. And since every measure one set is stationary, it follows that κ is Mahlo. □

Lemma 10. *If κ is measurable, then κ has the tree property.*

Proof. Let $j : V \to M$ be an elementary embedding with critical point κ, and $M^\kappa \subset M$. Let T be a tree of height κ and levels of size less than κ. Then in M, $j(T)$ is a tree of height $j(\kappa)$. Note that if $\alpha < \kappa$, then the α-th level of T, T_α has size less than κ. So $j(T_\alpha) = T_\alpha$. I.e. for all $\alpha < \kappa$, α-th level of $j(T)$ is the same as the α-th level of T.

Let $u \in j(T)$ be a node of level κ. Let b be the set of predecessors of u. Then b is a branch through $j(T)$ of order type κ. Also for any $x \in b$, for some α, x belongs to the α-th level of $j(T)$, i.e. $x \in T_\alpha$. Also by elementarity, b is linearly ordered and meets every level of T. It follows that b is an unbounded branch through T. □

As a corollary we get that measurable cardinals are weakly compact.